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A note on velocity, vorticity and helicity of inviscid fluid elements

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A brief description is given, with a new geometrical derivation, of the changes in velocity, vorticity and helicity of fluid elements and fluid volumes in inviscid flow. When a compact material volume V_b moves with a velocity $\boldsymbol{v}_b$ in a flow which at infinity has a velocity $\boldsymbol{U}_\infty$ and uniform vorticity $\boldsymbol{\Omega}$, it is shown that in general there is a net change ΔH_E in the integral of helicity H_E in the external region V_E *outside* the volume, i.e. $H_E = \int_{V_E} \boldsymbol{u}\cdot\boldsymbol{\omega}\, dV$ changes by ΔH_E, where $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are the velocity and vorticity fields. When the vorticity at infinity is weak (i.e. $|\boldsymbol{\Omega}| V_b^{\frac{1}{3}} \ll |\boldsymbol{U}_\infty - \boldsymbol{v}_b|$) and when $\boldsymbol{\Omega}$ is parallel to $\boldsymbol{v}_b$ and $\boldsymbol{U}_\infty$, the change in the external helicity integral, ΔH_E is proportional to the dipole strength of V_b. For the case of volumes with reflectional symmetry about an axis parallel to their direction of motion (e.g. axisymmetric volumes), $\Delta H_E = -((\boldsymbol{v}_b - \boldsymbol{U}_\infty)\cdot\boldsymbol{\Omega})\, V_b C_H$, where $C_H = \frac{1}{3}(1+C_M)$, and C_M is the added mass coefficient. So for a sphere moving along the axis of a pure rotating flow, $\Delta H_E = -\frac{1}{2} V_b\, (\boldsymbol{v}_b \cdot \boldsymbol{\Omega})$, which is negative. Larger values of the local helicity $(\boldsymbol{u}\cdot\boldsymbol{\omega})$ are generated by the flow around a volume when $(\boldsymbol{v}_b - \boldsymbol{U}_\infty) \wedge \boldsymbol{\Omega} \neq 0$, but for symmetric volumes there is no net contribution to ΔH_E if $(\boldsymbol{v}_b - \boldsymbol{U}_\infty) \wedge \boldsymbol{\Omega} = 0$. These results are used to develop some new physical concepts about helicity in turbulent flow, in particular concerning the helicity associated with eddy motions in rotating flows and the relative speed E_b of the boundary defining a region of turbulent flow moving into an adjacent region of weak or non-existent turbulence.

1. Introduction and review of vorticity and velocity of fluid elements

Recent research in turbulence has been greatly helped by studying the dynamics and motion of particular vortical regions of the flow. These deterministic studies tend to focus either on the large-scale vortices and their mutual interactions (Melander & Hussain 1990) or on how these vortices interact with the small-scale vorticity field (Hussain 1986). To make the best use of vortex dynamics for improving our understanding and interpreting the computation of vortex interactions, it is important to use as effectively as possible the basic results connecting the vorticity $\boldsymbol{\omega}$ and the velocity $\boldsymbol{u}$ of fluid elements as they move, and the conservation conditions involving the integral of helicity density $(\boldsymbol{u}\cdot\boldsymbol{\omega})$ (hereafter abbreviated to helicity) within material volumes as they move in the flow. In a number of recent papers, the distribution of helicity in turbulent flow has been discussed and computed (e.g. Levich & Tsinober 1983; Rogers & Moin 1987) but no use has been made of volume integrals of helicity. Numerical computations, such as these, of viscous flows at high Reynolds number, are usually interpreted in terms of concepts of vortex dynamics that were developed for inviscid flows. Although there are limitations in this idealization, helicity does provide useful ideas for analysing vortex dynamics and the mechanics of vortical flows.

This note has a review element, in which we give a new geometrical derivation of the old but neglected result linking the change of velocity of fluid elements to their displacements. This naturally leads on to the well-established results of Moffatt (1969). We derive a new result showing how a net contribution to the helicity integral is generated outside fluid volumes as they move through regions of fluid with background vorticity. As a specific case, a calculation is given for the helicity generated outside a spherical volume such as a bubble or a spherical vortex as it moves through a region of weakly rotating flow, as it is evident that the conservation condition for the helicity integral can also be applied to some part of a region of vortical flow if the flow elsewhere does not change. We show how this idea may be useful in analysing and computing the motion of limited lengths of long vortex tubes where they interact with other vortices, while the rest of the vortex tubes are unchanged, a result that is derived from the study of Berger & Field (1984).

Finally, we show how these concepts and results can be used to study the helicity in turbulent flow.

For inviscid rotational flows which are adiabatic and reversible, the vorticity $\boldsymbol{\omega}(\boldsymbol{x},t)$ and density $\rho(\boldsymbol{x},t)$ at a point $\boldsymbol{x}$ and time t are related to the vorticity and density of the same fluid element located at $\boldsymbol{a}$ at time t_0, by Cauchy's result expressed in suffix notation (see Batchelor 1967, p. 276) as

$$\frac{\omega_i(\boldsymbol{x},t)}{\rho(\boldsymbol{x},t)} = \frac{\omega_j(\boldsymbol{a},t_0)}{\rho(\boldsymbol{a},t_0)}\left(\frac{\partial x_i}{\partial a_j}\right). \tag{1.1}$$

The physical explanation is that

$$\frac{\boldsymbol{\omega}}{\rho} \propto \mathrm{d}\boldsymbol{l}, \tag{1.2}$$

where $\mathrm{d}\boldsymbol{l}$ is the (vector) length of a line element moving with the fluid, and chosen at time t_0 to be aligned with the local vorticity vector $\boldsymbol{\omega}(\boldsymbol{a},t_0)$.

There is also Weber's (1868) transformation (see Serrin 1959, p. 169) for the change in the velocity $\boldsymbol{u}$ of the fluid element, namely

$$u_i(\boldsymbol{x},t) = u_j(\boldsymbol{a},t_0)\frac{\partial a_j}{\partial x_i} + \frac{\partial \Psi(\boldsymbol{x},t)}{\partial x_i}, \tag{1.3a}$$

where $\Psi(\boldsymbol{x},t)$ is a scalar potential function of $\boldsymbol{x}$, which can be expressed in terms of the pressure field p and kinetic energy along the path of the fluid element from t_0 to t as an integral,

$$\Psi(\boldsymbol{x},t) = -\int_{t_0}^{t}\left[\int\frac{\mathrm{d}p}{\rho} - \tfrac{1}{2}u_k u_k\right]\mathrm{d}t. \tag{1.3b}$$

Equation (1.3a) has been rediscovered and proved several times, for example by Elsasser (1956) and most recently by Goldstein (1978).

Goldstein (1978) and Goldstein & Durbin (1980) showed that (1.3a) is an important practical result because it enables weak velocity perturbations $\boldsymbol{u}'$ to irrotational mean motions $\boldsymbol{U}(\boldsymbol{x})$ to be calculated directly. $\partial a_j/\partial x_i$ is calculated from the displacements (or drift functions) caused by the mean flow (for a review see Hunt

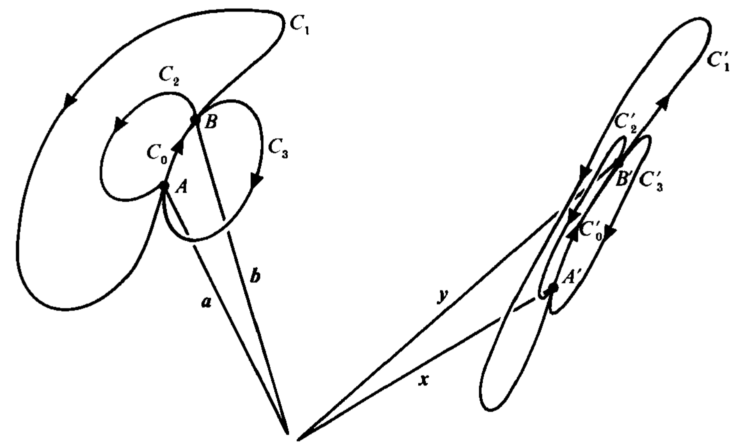


FIGURE 1. Showing how the material line C_0 between A and B moves to C'_0 between A' and B' and also how the 'partial' circuits $C_1, C_2, C_3 \ldots$ change to $C'_1, C'_2, C'_3 \ldots$. Kelvin's theorem is taken round C_0 and each partial circuit.

1987). Thence $u_i(t)$ is calculated in terms of $u_j(t_0)$. The scalar Ψ can be obtained from continuity.

There is a simple geometrical and physical demonstration of the validity of (1.3*a*), which we now give based on Kelvin's circulation theorem (see figure 1). Take the line integral $\int \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}$ along a path C_0 between the points A and B at time t_0. Then draw an arbitrary curve C_1 between B and A and take the line integral along this line C_1. There is now a line integral on a closed loop passing through A and B. We refer to C_0 and C_1 as 'partial' circuits of the whole loop. They are denoted in parentheses, e.g. (C_0), next to the integral.

After a time $(t-t_0)$ this loop has moved and been distorted. The points A and B have moved to A', B' and the partial circuits have moved to C'_0 and C'_1. By Kelvin's circulation theorem the circulation around C_0 and C_1 is equal to that around C'_0 and C'_1; so

$$\int_{A(C_0)}^{B} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t_0)+\int_{B(C_1)}^{A} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t_0)=\int_{A'(C'_0)}^{B'} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t)+\int_{B'(C'_1)}^{A'} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t). \quad (1.4)$$

Since we are mainly interested in how $\boldsymbol{u}(\boldsymbol{x})$ changes on the fluid element that travels from A to A' rather than in the line integral on the whole circuit, C_0+C_1, we might consider another possible 'partial' circuit C_2 from B to A, which changes to C'_2 at time t. In fact we can consider an infinite number of possible partial circuits C_n from B to A, which are changed to C'_n from B' to A' at time t. The circulation theorem (1.4) can be rewritten in terms of C_0 and any of these other partial circuits, i.e.

$$\int_{A'(C'_0)}^{B'} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t)-\int_{A(C_0)}^{B} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t_0)=\int_{B(C_n)}^{A} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t_0)-\int_{B'(C'_n)}^{A'} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t). \quad (1.5)$$

Since C_n or C'_n can be chosen arbitrarily, and since a line integral that is independent of the path is only a function of its end points, this difference can only be a scalar function Ψ of the positions $\boldsymbol{x}$, $\boldsymbol{y}$ of A' and B'. Therefore

$$\int_{A'}^{B'} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t)=\int_{A}^{B} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}(t_0)+\Psi(\boldsymbol{y})-\Psi(\boldsymbol{x}), \quad (1.6)$$

where $\Psi(\boldsymbol{x})$ is an unknown function of $\boldsymbol{x}$, t and t_0. It has the property that when $t = t_0$, $\Psi = 0$ for all $\boldsymbol{x}$.

Taking the limit of B' tending to A', and expressing the position of B' and B as $\boldsymbol{y} = \boldsymbol{x} + \mathrm{d}\boldsymbol{x}(t)$ and $\boldsymbol{b} = \boldsymbol{a} + \mathrm{d}\boldsymbol{a}(t_0) = \boldsymbol{a} + (\partial \boldsymbol{a}/\partial x_i)\,\mathrm{d}x_i$, (1.6) reduces to

$$u_i(\boldsymbol{x}, t)\,\mathrm{d}x_i(t) = u_j(\boldsymbol{a}, t_0)\,\mathrm{d}a_j(t_0) + \frac{\partial \Psi}{\partial x_k}\mathrm{d}x_k(t). \tag{1.7}$$

Then, since each component of $\mathrm{d}\boldsymbol{x}(t)$ can be taken as independent, (1.3) is recovered.

The potential Ψ is determined by assuming that the velocity field at time t satisfies continuity:

$$\frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho}\left[\frac{\partial}{\partial t} + u_n\frac{\partial}{\partial x_n}\right]\rho = \frac{\partial}{\partial x_i}\left[u_k(\boldsymbol{a}(\boldsymbol{x}), t_0)\frac{\partial a_k}{\partial x_i}\right] + \frac{\partial^2 \Psi}{\partial x_l^2}. \tag{1.8}$$

For incompressible flow, where $\partial u_i/\partial x_i = 0$, this equation is sufficient to define Ψ in terms of $\boldsymbol{u}(\boldsymbol{x}, t)$. If the density varies in the fluid, the results (1.1) and (1.3) are valid for each material element, provided that the fluid is barotropic (i.e. $\nabla p \wedge \nabla \rho = 0$) over the volume of the material element, and that any body forces per unit mass are conservative. In that case (1.8) shows that Ψ is a function of $\boldsymbol{u}$ and ρ.

A simple example of the insight provided by this elemental form of Kelvin's theorem is provided by the case of weak, small-scale, three-dimensional rotational fluctuations $\boldsymbol{u}$ in two-dimensional straining motions $\boldsymbol{U} = (-\alpha x_1, \alpha x_2, 0)$ where $\alpha > 0$.

Provided $\|\partial u_i/\partial x_j\|$ is much less than α, $\partial a_i/\partial x_j$ is determined by $\boldsymbol{U}$; so that $(\partial a_1/\partial x_1) = \mathrm{e}^{+\alpha(t-t_0)}$; $(\partial a_2/\partial x_2) = \mathrm{e}^{-\alpha(t-t_0)}$, $\partial a_3/\partial x_3 = 1$ and the cross-components are zero. Since Ψ is the solution of the Poisson's equation, and since $\partial \Psi/\partial x_i$ has components in all three directions, the magnitude of this gradient is less than $|u_j\,\partial a_j/\partial x_i|$. In general, for the directions in which $|\mathrm{d}l_i|$ has the greatest distortion, u_i^2 increases or decreases according to whether $|\mathrm{d}l_i|$ (or $|\partial x_i/\partial a_i|$) decreases or increases. For the other directions $\nabla\Psi$ may determine the change in u_i^2. So in this two-dimensional strain, u_1^2 increases and u_2^2 decreases because $|\mathrm{d}l_1|$ is decreased and $|\mathrm{d}l_2|$ is increased. But u_3^2 is unchanged by the term $\partial a_3/\partial x_3$ and is therefore determined by $\nabla\Psi$. Generally, as in this case, $\nabla\Psi$ acts to reduce the anisotropy, and therefore u_3^2 increases (Batchelor & Proudman 1954).

For three-dimensional fluctuations, qualitative vortex stretching arguments (e.g. Hunt 1973) are equally effective, but they do not apply to two-dimensional fluctuations for which the magnitude of vorticity is unchanged. In this example, if the initial fluctuations lie in the (x_1, x_2)-plane, their vorticity is not changed, but the distribution of vortex lines is altered as the vortex lines 'pile up' at the stagnation point (Hunt 1973, pp. 643, 672). By considering the changes in $\mathrm{d}l_i$, it follows that for two-dimensional as well as three-dimensional fluctuations, there is a reduction in u_2^2 and an increase in u_1^2.

An example of the physical insight given by (1.3) for inhomogeneous flow was given by Hunt (1987), along with a brief account of the informal proof presented here.

2. Helicity of fluid elements and fluid volumes

2.1. *General results*

The concept of the helicity density

$$h = \boldsymbol{u}\cdot\boldsymbol{\omega}(\boldsymbol{x}, t), \tag{2.1}$$

and the helicity integral

$$H = \int (\boldsymbol{u}\cdot\boldsymbol{\omega})\,\mathrm{d}V \tag{2.2}$$

were introduced to fluid mechanics by Moffatt (1969). He considered this integral over the whole flow or over a material volume bounded by a surface on which the normal component of vorticity is zero. Then H is conserved in inviscid flows for which ρ is a unique function of pressure p (i.e. the fluid is barotropic) and any body forces are conservative. Obviously, h is not a conserved property and in general varies in space and time in turbulent flc s. Note that if $h > 0$ or < 0 the motion has locally a right-handed or left-handed .elical structure.

Since then there have been many papers reporting on computations and measurements of h in various flows, but for H there have only been computational results (e.g. Kida & Murakami 1989). There has been particular interest in the mechanism for the generation and the distribution of helicity h in turbulent flows. In the coherent structures found in turbulent shear layers there are regions where h is large (both positive and negative) and other regions where h is very small (Hussain 1986). Levich & Tsinober (1983) and Rogers & Moin (1987) have computed the distribution of h in homogeneous turbulent flow. Since there can be no production of vorticity if $\boldsymbol{u} \wedge \boldsymbol{\omega} = 0$, it follows that the production of vorticity and the cascade of energy to small scales in turbulence must be very weak if $\boldsymbol{u}$ is parallel to $\boldsymbol{\omega}$ and $|h|$ becomes comparable with $(\overline{\boldsymbol{u}^2\boldsymbol{\omega}^2})^{\frac{1}{2}}$ (André & Lesieur 1977).

Almost no use has been made of the conservation condition that for inviscid flow $H =$ constant in the above discussions of helicity and turbulence, possibly because it has usually been interpreted as only being valid when the integral is taken over the whole region of the flow. (Moffatt's (1969) discussion of a blob of vorticity should be read carefully!). An important point of this note is to show how helicity integrals can and should be used within different regions of a flow. In §4 we consider in detail helicity in coherent structures.

From (1.7) and (1.1) the helicity density of a material element at time t can be expressed in terms of its value at time t_0. Since $\mathrm{d}\boldsymbol{l} \propto \boldsymbol{\omega}/\rho$

$$\begin{aligned}\frac{h(\boldsymbol{x},t)}{\rho} &= u_i(\boldsymbol{x},t)\left[\frac{\omega_i}{\rho}\right](\boldsymbol{x},t)\\ &= u_j(\boldsymbol{a},t_0)\left[\frac{\omega_j}{\rho}\right](\boldsymbol{a},t_0)+\left[\frac{\omega_k(\boldsymbol{x},t)}{\rho(\boldsymbol{x},t)}\right]\frac{\partial\Psi}{\partial x_k}.\end{aligned} \tag{2.3}$$

Therefore,
$$h(\boldsymbol{x},t) = \boldsymbol{u}\cdot\boldsymbol{\omega}(\boldsymbol{x},t) = \left[\frac{\rho(\boldsymbol{x},t)}{\rho(\boldsymbol{a},t_0)}\right]\boldsymbol{u}\cdot\boldsymbol{\omega}(\boldsymbol{a},t_0)+\boldsymbol{\omega}\cdot\nabla\Psi. \tag{2.4}$$

Note that $h(\boldsymbol{x},t)$ does not equal $h(\boldsymbol{a},t_0)$.

2.2. *Helicity of material volumes of fluid*

Now consider a set of non-overlapping volumes each of volume V_b, in each of which $\boldsymbol{\omega} \neq 0$. Let the surface around each volume be S_b, such that $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$, and let the flow within the space between these volumes be irrotational, i.e. $\boldsymbol{\omega} = 0$ (see figure 2*a*). The helicity density $h = \boldsymbol{u}\cdot\boldsymbol{\omega}$ can be integrated over each volume V_b to obtain the helicity integral H_b for each material volume.

Using (2.4), H_b at time t can be related to its value at time t_0, since

$$H_b(t) = \int_{V_b}\frac{\rho(\boldsymbol{x},t)}{\rho(\boldsymbol{a},t_0)}(\boldsymbol{u}\cdot\boldsymbol{\omega}(\boldsymbol{a},t_0))\,\mathrm{d}\boldsymbol{x}+\int_{V_b}\omega_k\frac{\partial\Psi}{\partial x_k}\mathrm{d}\boldsymbol{x}. \tag{2.5}$$

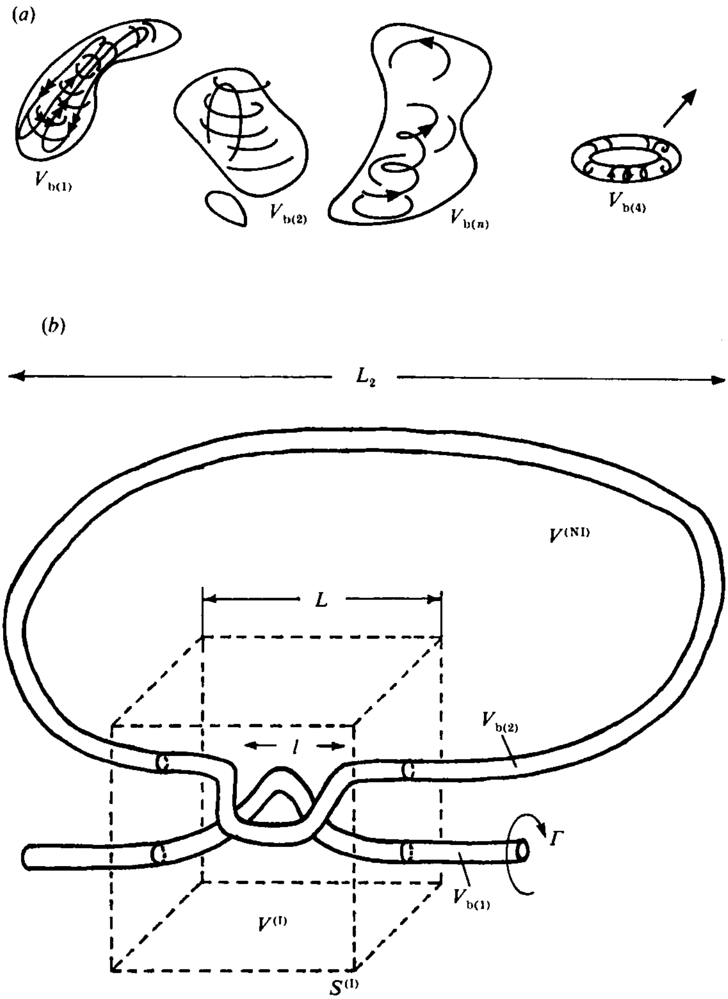


FIGURE 2. (a) Discrete non-overlapping volumes $V_{b(n)}$ of inviscid vortical flow separated by irrotational flow. The density of each $V_{b(n)}$ may differ but within $V_{b(n)}$ the flow is adiabatic and reversible. The helicity integral H_b is conserved for each volume. (b) Particular case of the interaction between two vortical flow volumes $V_{b(1)}$, $V_{b(2)}$ occurring in a small region with lengthscale l, to show how the conservation of the helicity integral in a volume $V^{(I)}$ can be used even when vortex lines cut the surface $S^{(I)}$ of $V^{(I)}$, provided (L/l) is large enough and the flow in $V^{(NI)}$ does not change.

From the conservation of mass of each material element

$$\rho(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \rho(\boldsymbol{a}, t_0)\,\mathrm{d}\boldsymbol{a},$$

and since $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$, (2.5) becomes

$$H_b(t) = \int_{V_b} (\boldsymbol{u}\cdot\boldsymbol{\omega}(\boldsymbol{a}, t_0))\,\mathrm{d}\boldsymbol{a} + \int_{S_b} \Psi(\boldsymbol{x}, t)\,\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S}, \tag{2.6}$$

where $\mathrm{d}\boldsymbol{S}$ is the outward normal area element on S_b. Since $(\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S}) = 0$ on S_b it follows that the helicity integral for each material volume as it moves around is constant, i.e.

$$H_b(t) = H_b(t_0). \tag{2.7}$$

It is simpler to use the Eulerian equation for $\boldsymbol{u}$ and $\boldsymbol{\omega}$ to calculate the rate of change of the helicity integral $H(t)$ within a volume *fixed* in space. It is found that for a barotropic flow

$$\frac{\partial H(t)}{\partial t} = -\int_{S_V} \left\{ (\boldsymbol{u}\cdot\hat{\boldsymbol{n}}(\boldsymbol{x},t))\,(\boldsymbol{u}\cdot\boldsymbol{\omega})\,(\boldsymbol{x},t) + (\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}}) \left[\frac{p}{\rho} - \frac{u^2}{2} \right] \right\} \mathrm{d}S_V. \tag{2.8}$$

Hence if $|\boldsymbol{\omega}| = 0$, on S_V, $\partial H/\partial t = 0$. If S_V is outside V_b, (2.8) is consistent with (2.7).

In turbulent flows the total helicity integral H taken over the whole volume is usually small, but over any particular regions of the flow where the vorticity is large, H_b and H may be significant (see §4, where we define 'small' and 'significant'). In some shear flows, H_b may be non-zero and have the same sign for most of the vortical eddies (e.g. the large swirling ring vortices in swirling turbulent jets).

The result (2.7) shows that in the absence of viscous diffusion and any vorticity connecting the different volumes, the helicity integral H_b of each volume is conserved, even though the interaction between the volumes may be strong, causing large changes of $\boldsymbol{u}$ and $\boldsymbol{\omega}$ within the volumes. Also in the presence of any large-scale irrotational motion H_b is conserved. Note that in such interactions, the impulse $\boldsymbol{P}_\mathrm{b} = \frac{1}{2}\rho \int (\boldsymbol{x} \wedge \boldsymbol{\omega})\,\mathrm{d}V$ of each volume is generally changed (Hunt 1987).

In some interactions between non-overlapping regions of flow occupying material volumes V_1 and V_2, with lengthscales L_1 and L_2, the interactions only lead to changes in the vortical field over a limited region of length l which is much less than L_1 and L_2. In that case the helicity integrals H_b for each of the volumes $V_{\mathrm{b}(1)}$ and $V_{\mathrm{b}(2)}$ can be expressed as the sum of two integrals, $H_\mathrm{b}^{(\mathrm{I})}$ over the regions $V^{(\mathrm{I})}$ where the interaction takes place and $H_\mathrm{b}^{(\mathrm{NI})}$ over the rest of the space $V^{(\mathrm{NI})}$ where they do not take place. Thus

$$H_\mathrm{b}(t) = H_\mathrm{b}^{(\mathrm{I})}(t) + H_\mathrm{b}^{(\mathrm{NI})}(t). \tag{2.9}$$

Since H_b is unchanged and within the volume $V^{(\mathrm{NI})}$, $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are undisturbed, it follows that the helicity integral over a partial fluid volume $H_\mathrm{b}^{(\mathrm{I})}$ is unchanged. So in some circumstances during the interaction, the helicity integral is conserved over a *partial* fluid volume.

One interesting consequence of this result is that it is impossible for a propagating helical disturbance to be generated on part of a vortex tube by any irrotational flow field around it, such as may be caused by another vortex tube. By 'propagating' is meant a disturbance that travels far from where it is generated on the vortex tube.

This result can be applied to computational studies of local interactions on scale l between parts of vortex tubes of strength Γ where the computational domain extends over a scale $L \gg l$, (see figure 2*b*). (See also Berger & Field 1984.) From (2.8) the rate of change of $H_\mathrm{b}^{(\mathrm{I})}$ within the interaction domain can be expressed in terms of the initial values and perturbations (denoted by $\boldsymbol{u}^{(0)}$ and $\Delta\boldsymbol{u}$) to the velocity, vorticity and pressure at the surface of the domain $S^{(\mathrm{I})}$. For the case of vortex tubes, this surface can be chosen so that $\boldsymbol{u}^{(0)}\cdot\hat{\boldsymbol{n}} = 0$ over $S^{(\mathrm{I})}$ and $\boldsymbol{\omega}^{(0)}$ is the same at the two intersections. Then

$$\frac{\partial H_\mathrm{b}^{(\mathrm{I})}}{\partial t} = -\int_{S^{(\mathrm{I})}} \left\{ (\Delta\boldsymbol{u}\cdot\hat{\boldsymbol{n}})\,(\Delta\boldsymbol{u}\cdot\boldsymbol{\omega}^{(0)} + \boldsymbol{u}^{(0)}\cdot\Delta\boldsymbol{\omega}) + (\Delta\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}}) \left[\frac{p^{(0)}}{\rho} - \tfrac{1}{2}(u^{(0)})^2 \right] \right.$$
$$\left. + (\boldsymbol{\omega}^{(0)}\cdot\hat{\boldsymbol{n}}) \left[\frac{\Delta p}{\rho} - (\boldsymbol{u}^{(0)}\cdot\Delta\boldsymbol{u} + \tfrac{1}{2}(\Delta u)^2) \right] \right\} \mathrm{d}S. \tag{2.10}$$

The magnitude of the velocity perturbations Δu induced by the local distortions of the vortex tubes on a scale l at time $t = 0$ is equivalent to that produced by a dipole of strength Γl^2. So over the surface $S^{(1)}$, which is at a distance of order L from the interaction zone,

$$u^{(0)}\,(=|\boldsymbol{u}^0|) \sim \frac{\Gamma}{l}, \quad \omega^{(0)}\,(=|\boldsymbol{\omega}^0|) \sim \frac{\Gamma}{l^2}, \quad \Delta u \sim \frac{\Gamma l^2}{L^3}, \quad \frac{\Delta p}{\rho} \sim \frac{l^4\Gamma^2}{L^6}.$$

But the perturbation vorticity on $S^{(1)}$ changes with time:

$$\Delta\omega \sim t\left(\frac{\Delta u}{L}\right)\omega^{(0)} \sim \frac{t\Gamma^2}{L^4}.$$

Thence, from (2.10)

$$\frac{\partial H_{\mathrm{b}}^{(1)}}{\partial t} = O\left(\frac{\Gamma^3 l^4}{L^6}, \frac{\Gamma^3 l}{L^3}\frac{t\Gamma}{lL}\right). \tag{2.11}$$

This estimate indicates that when a helicity integral is evaluated over a finite computational domain, its size does not have to be very large compared with the lengthscales over which the interactions occur, in order that the computed integrals will be conserved, for each fluid volume ($V_{\mathrm{b}(1)}, V_{\mathrm{b}(2)}$).

The result (2.7) can be applied to flows with variation in density in space and time, provided that the fluid is barotropic within each volume and that any body forces per unit mass are conservative. In turbulent flows with significant density fluctuations it is customary to use Favre-averaging (i.e. considering $\overline{\rho u_i u_i}$ as opposed to Reynolds-averaging (i.e. $\overline{u_i u_j}$)) in analysing the dynamics (Favre 1969). The results (2.4)–(2.8) clearly suggest that, when considering helicity, it is possible to construct integrals that are invariant and independent of barotropic density variations.

3. Helicity induced outside fluid volumes in rotational flows

We consider the distribution of helicity in *steady inviscid rotational* flows *outside* compact material volumes which are moving through the flow. From this idealized flow some interesting suggestions emerge about helicity in turbulent flows. Consider a large domain of fluid in solid-body rotation with vorticity $\boldsymbol{\Omega}$, into which a volume V_{b} is introduced, moving steadily with a velocity $\boldsymbol{v}_{\mathrm{b}}$. The volume might be a bubble or the inside of a closed streamline surface of a vortical eddy such as a vortex ring. There may or may not be a vortical flow within the volume (figure 3*a*). Let the diameter of the volume be $2a$, which is of the order $(V_{\mathrm{b}})^{\frac{1}{3}}$.

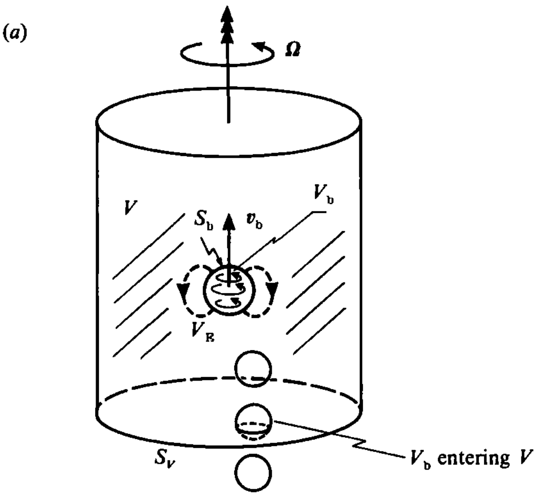
The solution makes use of a Galilean transformation of the coordinate system to calculate $\boldsymbol{u}$ and $\boldsymbol{\omega}$, but since helicity is not a Galilean-invariant quantity, helicity and its integral are calculated in a fixed coordinate system.

In order to obtain analytical solutions, we consider situations where the background vorticity is weak compared with the strain rate $|\boldsymbol{v}_{\mathrm{b}}|/a$ in the flow around the volume. Also, this enables us to obtain solutions for spherical volumes moving at arbitrary angles to the flow. First we consider volumes with arbitrary shape and then those with certain symmetries.

3.1. *The volume moves parallel to the rotation axis*

The formal problem to be solved is to obtain the solution of

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}, \tag{3.1}$$



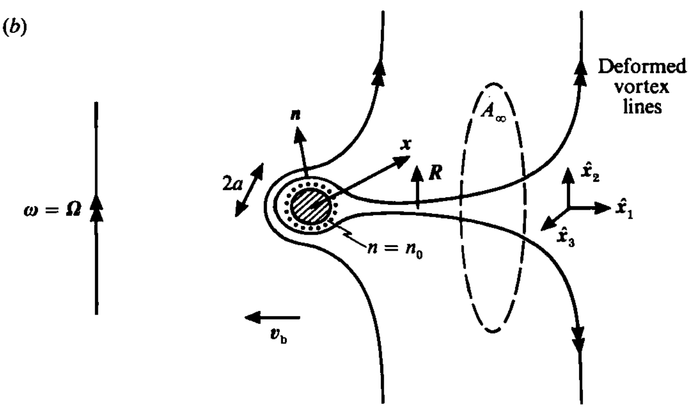


FIGURE 3. (*a*) Volume V_b moving with velocity $\boldsymbol{v}_b$ parallel to a rotating flow. V_E is the external region round the volume. V is the total volume. S_V is the surface of the total volume. (*b*) Volume V_b moving with velocity $\boldsymbol{v}_b$ perpendicular to $\boldsymbol{\Omega}$. Note how as the vortex lines are distorted, they remain parallel to the surfaces where the 'drift' or time function $T(\boldsymbol{x})$ is constant.

where $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ subject to $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \boldsymbol{v}_b \cdot \hat{\boldsymbol{n}}$ on S_b, where the unit normal unit vector $\boldsymbol{n} = 0$ and $\hat{\boldsymbol{n}}$ is the normal vector. Far from the volume, as $|\boldsymbol{n}|/a \rightarrow \infty$, $\boldsymbol{u} \rightarrow \frac{1}{2}\boldsymbol{\Omega} \wedge \boldsymbol{x}$. The solution for $\boldsymbol{\omega}$ in terms of $\boldsymbol{u}$ is obtained by rewriting (3.1) in coordinates $\boldsymbol{x}'$ moving in the body, where $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}_b t$, and where $\boldsymbol{x} = \boldsymbol{v}_b t$ is the location of a reference point (e.g. the centre) of the volume. Then, in terms of the relative velocity,

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}_b. \tag{3.2a}$$

Since the flow in these coordinates is steady, (3.1) becomes

$$(\boldsymbol{u}' \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}', \tag{3.2b}$$

where

$$\boldsymbol{u}' \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \boldsymbol{n} = 0, \tag{3.2c}$$

and

$$\left.\begin{aligned} &\boldsymbol{u}' \rightarrow (\tfrac{1}{2}\boldsymbol{\Omega} \wedge \boldsymbol{x}) - \boldsymbol{v}_b \\ &\boldsymbol{\omega} = \boldsymbol{\Omega} \end{aligned}\right\} \text{ as } \quad |\boldsymbol{n}|/a \rightarrow \infty. \tag{3.2d}$$

We consider the solution where the background rotation is weak, compared with the gradient in the velocity field around the moving volume, i.e.

$$\epsilon = \frac{a\Omega}{v_b} \ll 1,$$

where
$$v_b = |\boldsymbol{v}_b|, \quad \Omega = |\boldsymbol{\Omega}|. \tag{3.3}$$

For future reference we define a unit velocity vector,

$$\hat{\boldsymbol{v}}_b = \frac{\boldsymbol{v}_b}{v_b}. \tag{3.4}$$

The solutions for $\boldsymbol{u}', \boldsymbol{\omega}'$ can be expressed as expansions in ϵ, normalized on v_b and a:

$$\left.\begin{aligned} \boldsymbol{u}' &= v_b(\boldsymbol{u}'^{(0)} + \epsilon \boldsymbol{u}'^{(1)} + \dots) \\ \boldsymbol{\omega}' &= \frac{v_b}{a}(\boldsymbol{\omega}'^{(0)} + \epsilon \boldsymbol{\omega}'^{(1)} + \dots) \end{aligned}\right\}, \tag{3.5}$$

where, as $|\boldsymbol{n}|/a \rightarrow \infty$,

$$\boldsymbol{u}'^{(0)} \rightarrow -\hat{\boldsymbol{v}}_b, \quad \boldsymbol{u}'^{(1)} \rightarrow \frac{1}{2}\frac{\boldsymbol{\Omega} \wedge \boldsymbol{x}}{a\Omega} \tag{3.6a}$$

and
$$\boldsymbol{\omega}'^{(0)} = 0, \quad \boldsymbol{\omega}'^{(1)} = \boldsymbol{\Omega}/\Omega. \tag{3.6b}$$

Therefore, from (3.2*b*) $\boldsymbol{\omega}'^{(0)}$ is zero everywhere and $\boldsymbol{u}'^{(0)}$ is the unique irrotational velocity field, with boundary conditions (3.2*c*) and (3.6*a*). The first-order vorticity satisfies

$$(\boldsymbol{u}'^{(0)} \cdot \nabla)\, \boldsymbol{\omega}'^{(1)} = (\boldsymbol{\omega}'^{(1)} \cdot \nabla)\, \boldsymbol{u}'^{(0)}. \tag{3.7}$$

Since $\boldsymbol{\omega}'^{(1)}$ is parallel to $\boldsymbol{u}'^{(0)}$ as $|\boldsymbol{n}| \rightarrow \infty$, (from 3.6*a*, *b*) it follows that the solution to (3.7) is

$$\boldsymbol{\omega}'^{(1)}(\boldsymbol{x}) = \alpha \boldsymbol{u}'^{(0)}(\boldsymbol{x}) \quad \forall \boldsymbol{x}'(|\boldsymbol{n}| \geqslant 0), \tag{3.8}$$

where
$$\alpha = -\frac{\boldsymbol{\Omega} \cdot \boldsymbol{v}_b}{(v_b \Omega)}. \tag{3.9}$$

(This is a well-known solution; see the review by Hunt 1987.) Note that to leading order the vorticity is parallel to its surface on the volume (i.e. $\hat{\boldsymbol{n}} \cdot \boldsymbol{\omega} = 0$ on $|\boldsymbol{n}| = 0$).). Thence from (3.6*a*), (3.8), (3.9) the helicity is given as

$$h = (\boldsymbol{u} \cdot \boldsymbol{\omega}) = v_b \frac{v_b}{a}[(\boldsymbol{u}'^{(0)} + \hat{\boldsymbol{v}}_b) \cdot \epsilon \alpha \boldsymbol{u}'^{(0)}] = -(\boldsymbol{\Omega} \cdot \boldsymbol{v}_b)\,(\boldsymbol{u}'^{(0)} + \boldsymbol{v}_b) \cdot \boldsymbol{u}'^{(0)}. \tag{3.10}$$

Inspection of (3.10) shows how the sign of h varies in the flow around a typical volume. Consider a spheroid with one axis aligned with the flow. The component of the potential velocity $\boldsymbol{u}'^{(0)}$ parallel to $\hat{\boldsymbol{v}}_b$ decreases from 0 at the stagnation point to less than -1 at the sides of the volume. Here the component of velocity $(\boldsymbol{u}'^{(0)} + \hat{\boldsymbol{v}}_b) \cdot \hat{\boldsymbol{v}}_b$ in fixed coordinates is negative. Therefore around the sides of these bodies h is negative if $\boldsymbol{\Omega} \cdot \boldsymbol{v}_b$ is positive. On the other hand, on the stagnation streamline the component of $\boldsymbol{u}'^{(0)}$ parallel to $\hat{\boldsymbol{v}}_b$ lies between -1 and 0, so that in fixed coordinates in this direction $(\boldsymbol{u}'^{(0)} + \hat{\boldsymbol{v}}_b) \cdot \hat{\boldsymbol{v}}_b$ is positive. Therefore on the stagnation streamline, $(\boldsymbol{u}'^{(0)} + \hat{\boldsymbol{v}}_b) \cdot \boldsymbol{u}'^{(0)}$ is negative and h is positive if $\boldsymbol{\Omega} \cdot \boldsymbol{v}_b$ is positive (figure 3*a*).

The external helicity integral H_E (which is dimensional and defined in fixed coordinates) over the external volume V_E can be calculated from (3.10), for general flows satisfying (3.3). Since $\boldsymbol{u}'^{(0)}$ is irrotational, we can express $\boldsymbol{u}'^{(0)}+\hat{\boldsymbol{v}}_b$ in terms of a normalized potential ϕ as

$$\boldsymbol{u}'^{(0)}+\hat{\boldsymbol{v}}_b = \nabla\phi, \tag{3.11}$$

where $\nabla^2\phi = 0$, and from (3.10)

$$H_E = \int_{V_E} h\,dV = -(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)\int_{V_E}[\nabla\phi\cdot\nabla\phi-\nabla\cdot(\phi\hat{\boldsymbol{v}}_b)]\,dV$$

$$= -(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)a\left\{\int_{S_\infty}[\phi\nabla\phi-\phi\hat{\boldsymbol{v}}_b]\cdot\hat{\boldsymbol{n}}\,dS-\int_{S_b}[\phi\nabla\phi-\phi\hat{\boldsymbol{v}}_b]\cdot\hat{\boldsymbol{n}}\,dS\right\}, \tag{3.12a}$$

where S_∞ is an arbitrary surface far from V_b.

Since $(\nabla\phi\cdot\hat{\boldsymbol{n}}) = (\hat{\boldsymbol{v}}_b\cdot\hat{\boldsymbol{n}})$ on S_b, and $\nabla\phi\rightarrow O(r^{-3})$ as $r\rightarrow\infty$,

$$H_E = (\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)a\int_{S_\infty}(\hat{\boldsymbol{v}}_b\cdot\hat{\boldsymbol{n}})\,\phi\,dS \quad \text{as} \quad r\rightarrow\infty. \tag{3.12b}$$

Following Batchelor (1967, p. 399) as $r\rightarrow\infty$, $\phi \sim -C_1\cos\theta/r^2$ where $\cos\theta = \hat{\boldsymbol{v}}_b\cdot\hat{\boldsymbol{n}}$, and $dS = 2\pi r^2\sin\theta\,d\theta$,

$$H_E = -\tfrac{4}{3}\pi C_1\,V_b(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)$$
$$= -\tfrac{4}{3}\pi C_1(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)\,V_b.$$

Since $\hat{\boldsymbol{v}}_b$ and ϕ are normalized, C_1 is a constant which depends only on the shape of the volume. C_1 and therefore H_E are defined by the strength of the equivalent dipole that produces the same perturbation in the far field as the volume. Up to this point the analysis applies to any shape and C_1 can always be computed.

For the class of shapes that have reflectional symmetry about an axis parallel to $\boldsymbol{v}_b$, the dipole is aligned with the flow. Then the integral over the surface at infinity (3.12*b*) can be expressed in terms of the kinetic energy of the velocity field *outside* the volume ($\frac{1}{2}C_M\rho v_b^2\,V_b$), where C_M is the added mass coefficient, and of the kinetic energy of the fictional flow within the volume ($\frac{1}{2}\rho v_b^2\,V_b$), so that

$$C_1 = \frac{1}{4\pi}(1+C_M), \tag{3.12c}$$

where
$$C_M = \frac{1}{V_b}\int_{V_E}(\nabla\phi)^2\,dV = -\frac{a}{V_b}\int_{S_b}\phi\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{v}}_b\,dS$$

(Batchelor 1967, pp. 398–403). Then (3.12*b*) becomes

$$H_E = -C_H(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)V_b, \quad \text{where} \quad C_H = \tfrac{1}{3}(1+C_M). \tag{3.12d}$$

One implication of (3.12*d*) is that any elongated cylindrical shape (e.g. a needle) moving parallel to itself produces a negligible change in kinetic energy (i.e. $C_M = 0$), but because it displaces the flow, it induces a change in helicity, with the coefficient $C_H = \frac{1}{3}$.

However, for volumes that significantly disturb the flow, C_M and therefore C_H are increased. For a sphere, $C_M = \frac{1}{2}$, $C_H = \frac{1}{2}$; for a circular cylinder moving normal to its axis, $C_M = 1$, $C_H = \frac{2}{3}$, and for a disc moving normal to its axis, the high kinetic energy at the edges leads to $C_M = \frac{8}{3}$, and thence $C_H = \frac{11}{9}$.

Thus, from (3.10), and (3.12) any symmetrical fluid volume moving with velocity $\boldsymbol{v}_b$ parallel to a weak rotational motion produces a net helicity integral

$$H = -(\boldsymbol{\Omega}\cdot\boldsymbol{v}_b)\,V_b\,C_H+H_b, \tag{3.13}$$

where H_b is the helicity integral within the volume.

If there is a mean uniform motion $\boldsymbol{U}_\infty$ parallel to $\boldsymbol{\Omega}$, the helicity density is non-zero far from the volume, and the integral (3.12*a*) does not converge. However, consider the change in the net helicity integral ΔH_E produced by the volume which is defined by

$$\Delta H_E = \int_{V_E} (\boldsymbol{\omega}\cdot\boldsymbol{u} - \boldsymbol{\Omega}\cdot\boldsymbol{U}_\infty)\,\mathrm{d}V.$$

(Although this is equivalent to a change in the frame of reference, it is not generally possible to assume how H changes under a Galilean transformation without a detailed calculation, especially when, as in this case, $\int(\boldsymbol{\omega}-\boldsymbol{\Omega})\,\mathrm{d}V$ is a divergent integral.) Assuming that the vorticity is weak enough that

$$|\boldsymbol{v}_b - \boldsymbol{U}_\infty| \gg |\boldsymbol{\Omega}|\,a. \tag{3.14}$$

then the same analysis as in (3.12) can be used, whence

$$\Delta H_E = ((\boldsymbol{U}_\infty - \boldsymbol{v}_b)\cdot\boldsymbol{\Omega})\,V_b\,C_H. \tag{3.15}$$

Note that (3.12) and (3.15) do *not* contradict the result that the helicity integral H is constant in inviscid flow.

If we consider a finite volume V of fluid in rotation with $U_\infty = 0$, then as each volume V_b enters it, the integral H in V is increased by ΔH_E (from Moffatt 1969), where ΔH_E is determined by the integral in (2.8) involving the flow across the entering surface S_V (figure 3*a*).

In general it is simpler to estimate H from the local motion rather than from the production of H by motions across the boundary or by viscous processes, so we concentrate on the former direct approach.

3.2. *Volumes moving perpendicular to the rotation axis*

If in our model flow, the volume V_b moves with velocity $\boldsymbol{v}_b = (v_b, 0, 0)$ perpendicular to the imposed velocity $\boldsymbol{\Omega} = (0, \Omega, 0)$, then the vorticity and velocity outside V_b are distorted. By again assuming weak vorticity, i.e. $|\Omega| \ll v_b/a$, it is possible to use the result (1.1) (with the methods reviewed by Hunt 1987).

Once again we analyse the change in $\boldsymbol{\omega}$ by considering a steady flow with a steady incident velocity $(-\boldsymbol{v}_b)$ in a frame of reference moving with the volume. The weak vorticity is transported and distorted by the potential flow $\boldsymbol{u}'^{(0)}$ around the volume. Since $\boldsymbol{\omega}$ is initially perpendicular to the approach flow $(-\boldsymbol{v}_b)$ it lies in surfaces where the 'drift' or 'time of flight' $T(\boldsymbol{x})$ (of the potential flow) is constant. These drift surfaces are material surfaces which are deformed as they are transported over the volume. Consequently the vortex lines remain within the surfaces and are therefore perpendicular to $\boldsymbol{\nabla}T$ (figure 3*b*). Using the theory originated by Lighthill (1956), and later applied by Durbin (1981) and Hunt (1987), and using the fact that $\boldsymbol{\nabla}T\cdot\boldsymbol{u}'^{(0)} = 1$ (in normalized form), $\boldsymbol{\omega}$ can be expressed in terms of $\boldsymbol{\nabla}T$ and $\boldsymbol{u}'^{(0)}$ (to first order). Then the helicity can be expressed in terms of the vector $\boldsymbol{x}$ from the volume centre and the radius vector $\hat{\boldsymbol{R}}$ from the axis of motion, namely

$$\boldsymbol{\omega}\cdot\boldsymbol{u} = -(\boldsymbol{\nabla}T \wedge (\hat{\boldsymbol{x}}_1 \wedge \hat{\boldsymbol{x}})\cos\theta\cdot\boldsymbol{u}'^{(0)})\,v_b\,\Omega\hat{R}_0/\hat{R}, \tag{3.16}$$

where $\cos\theta = \hat{\boldsymbol{R}}\cdot\boldsymbol{\Omega}/\Omega$, $\hat{\boldsymbol{R}} = -(\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{x}}_1) \wedge \hat{\boldsymbol{x}}_1 = \hat{\boldsymbol{x}} - (\hat{\boldsymbol{x}}_1\cdot\hat{\boldsymbol{x}})\,\hat{\boldsymbol{x}}_1$, and $\hat{\boldsymbol{x}}, \boldsymbol{x}_1$ are unit vectors in the $\boldsymbol{x}$ and $\boldsymbol{x}_1$ directions. Note $\hat{\boldsymbol{x}}_1 = -\hat{\boldsymbol{v}}_b$ and $\hat{R} \to \hat{R}_0$ as $\hat{\boldsymbol{x}} \to \infty$.

Far downstream of the volume the drift surfaces, and therefore $\boldsymbol{\omega}$, become nearly parallel to the x-axis (assumed to be the axis of symmetry) (Hawthorne & Martin

1955). But the large component parallel to the x-axis, ω_1, has opposite signs above and below the plane $y = 0$ (or $\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}_1 \gtrless 0$ for $\boldsymbol{x}\cdot\boldsymbol{\Omega} \lesseqgtr 0$, respectively). The maximum positive and negative values of helicity above and below the $y = 0$ plane are given by

$$h \approx -v_b(\nabla T\cdot\boldsymbol{\Omega}). \tag{3.17}$$

Since any fluid element approaching V_b on the stagnation line takes an infinite drift time T to reach the x-axis downwind of V_b, $T(R)$ and $\nabla T(R)$ tend to infinity as the distance R from the x-axis tends to zero. Therefore from (3.17) strong helicity variations are generated near the axis of the volume (and on its surface), and they persist far downstream of the volume. In the previous case of $\boldsymbol{v}_b$ parallel to $\boldsymbol{\Omega}$, there was no effect on h far downstream of the volume.

The next step is to consider the contribution to the net helicity integral H_E. From (3.17) it follows that over the area (e.g. A_∞) of any plane downwind of V_b, the integral of helicity is zero. Therefore to evaluate the integral of helicity around the volume H_E, we need only consider the integral near the volume. The component of vorticity parallel to the surface becomes very large as the vortex lines are wrapped around the surface.

Near the surface of V_b, where the distance normal to the surface $n = |\boldsymbol{n}|$ is small compared with a, the helicity (from (3.16)) is

$$h = \boldsymbol{\omega}\cdot\boldsymbol{u} \approx O((\nabla T\cdot\hat{\boldsymbol{n}})\,\Omega v_b) \propto \Omega a v_b/n. \tag{3.18}$$

Because $h \propto (1/n)$ as $n \to 0$, H_E is a divergent volume integral. Suppose the integral is taken to a small distance $n_0\,(\ll a)$ from the surface of V_b, then from the symmetry it is clear that $H_E(n > n_0) = \int_{V_E}(\boldsymbol{\omega}\cdot\boldsymbol{u})\,\mathrm{d}V$ can only be non-zero if the flow around V_b is asymmetric. (In a viscous flow, such singularities would be smoothed out, and the integral would converge.)

Since $H_E = 0$ for a sphere moving perpendicular to weak vorticity it follows that the change in external helicity ΔH_E can be computed for a sphere moving with a velocity $\boldsymbol{v}_b$ in a general flow field $\boldsymbol{U}_\infty$ with vorticity $\boldsymbol{\Omega}$ provided that (3.14) is satisfied and $a \ll (|\boldsymbol{U}_\infty - \boldsymbol{v}_b|/\|\nabla \boldsymbol{U}_\infty\|)$. The result is the same as (3.15), namely

$$\Delta H_E = \tfrac{1}{2}((\boldsymbol{U}_\infty - \boldsymbol{v}_b)\cdot\boldsymbol{\Omega})\,V_b. \tag{3.19}$$

4. Aspects of helicity in turbulent flows

4.1. *Helicity statistics*

In this section we consider helicity in turbulent flows. Any large-scale rotation or acceleration is assumed to be weak enough not to affect the turbulence. We distinguish between the contributions to the helicity density h from the mean velocity and vorticity $\bar{\boldsymbol{u}}, \bar{\boldsymbol{\omega}}$ and from all kinds of random fluctuations $\boldsymbol{u}_r, \boldsymbol{\omega}_r$.

It can also be revealing to focus on the large-scale, slowly changing but randomly moving coherent structures in turbulent flows. In these flow regions we distinguish between the large-scale coherent velocity and vorticity fields, defined in a frame moving with the structure, i.e. $\boldsymbol{u}_c, \boldsymbol{\omega}_c$, which are correlated across the structure, and the incoherent random fields $\boldsymbol{u}_{rc}, \boldsymbol{\omega}_{rc}$, which include small-scale motions and (at high Reynolds number) the motions with highest vorticity (Hussain 1986). Thus $\boldsymbol{u} = \boldsymbol{u}_c + \boldsymbol{u}_{rc}$. Note that in a fixed frame, where $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}_r$, the mean fields $\bar{\boldsymbol{u}}, \bar{\boldsymbol{\omega}}$ are largely determined by $\boldsymbol{u}_c, \boldsymbol{\omega}_c$, but the random fields $\boldsymbol{u}_r, \boldsymbol{\omega}_r$, have contributions from the coherent and incoherent fields.

Since the helicity is quadratic, the mean helicity $\bar{h}$, and for coherent structures, the coherent helicity h_c, may have contributions from both the random and incoherent fields, respectively. Thus in the fixed frame

$$\bar{h} = \bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{\omega}} + \overline{h_r}, \quad \text{where} \quad \overline{h_r} = \overline{\boldsymbol{u}_r\cdot\boldsymbol{\omega}_r}, \tag{4.1}$$

and in the frame of reference of a coherent structure

$$h_c = \boldsymbol{u}_c\cdot\boldsymbol{\omega}_c + \{h_{rc}\}, \quad \text{where} \quad h_{rc} = \boldsymbol{u}_{rc}\cdot\boldsymbol{\omega}_{rc}, \tag{4.2}$$

where { } denotes averaging over an ensemble of similar coherent structures.

The r.m.s. of the fluctuations of h_r and h_{rc} about their mean values are also of interest, as defined by

$$h'_r = [\overline{(h_r - \bar{h}_r)^2}]^{\frac{1}{2}}, \quad h'_{rc} = \{(h_{rc} - \{h_{rc}\})^2\}^{\frac{1}{2}}. \tag{4.3}$$

Measures for the relative magnitude of the mean and r.m.s. values of helicity fluctuations are

$$\hat{\bar{h}}_r = \bar{h}_r/(u'_r\,\omega'_r), \quad \hat{h}'_r = h'_r/(u'_r\,\omega'_r) \tag{4.4a}$$

where $u'_r = (\overline{u_r^2})^{\frac{1}{2}}$ and $\omega'_r = (\overline{\omega_r^2})^{\frac{1}{2}}$. When $\hat{h}'_r$ is comparable with unity but $\hat{\bar{h}}_r$ is much less than one, it means that $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are aligned; but they are not on average in the same direction. Note that if the angle between the vectors $\boldsymbol{u}_r$ and $\boldsymbol{\omega}_r$ is θ, in isotropic turbulence $\overline{\cos^2\theta} = \hat{h}_r'^2 = \frac{1}{3}$ (Rogers & Moin 1987). But in the normalization of (4.4*a*), h'_r is divided by the smallest scales of motion which contribute most strongly to $\overline{\omega_r^2}$. Since in high-Reynolds-number turbulence the velocity and vorticity fluctuations are weakly correlated (Batchelor 1953), the mean and r.m.s. helicity fluctuation must be associated with the large-scale motion. Consequently a more revealing measure of helicity fluctuation would be a normalization based on characteristic values of the large scales, i.e.

$$\tilde{h}'_r = h'_r L_0/u_0'^2 \quad (> \hat{h}'_r), \tag{4.4b}$$

where u'_0 the r.m.s. velocity fluctuation and L_0 is a local integral scale of fluctuation.

From the idealized calculation of §3, we can estimate the possible magnitude of these helicity statistics for a flow field of distinct eddies (denoted by 't' for 'tourbillons') moving randomly through a mean flow with *weak* vorticity $\boldsymbol{\Omega} = (\Omega, 0, 0)$ (figure 4*a*). These eddies are analogous to vortex rings which propel themselves and have an approximately closed surface of streamlines around them in a frame of reference moving with the eddy. These large eddies provide most of the energy of the random components $\boldsymbol{u}_r, \boldsymbol{\omega}_r, h_r, h'_r$ of the whole flow. They are localized volumes with a net motion in a particular direction producing a positive high velocity within the volume and upstream and downstream of the volume, but a negative velocity outside them at their 'equator'. In a frame moving with the mean velocity, there must be a slow net backflow between the eddies.

If the proportion of the total volume (or 'void fraction') occupied by 'eddies' is Λ, and the eddies (or 'tourbillons') move with velocity v_t, the average values of u_1^3 and u_1^2 over the whole fluid can be estimated in terms of v_t and Λ as

$$\overline{u_1^3} \approx \Lambda v_t^3, \quad \overline{u_1^2} \approx \Lambda v_t^2. \tag{4.5a, b}$$

Therefore Λ can be estimated from the ratio

$$(\overline{u_1^3})^2/(\overline{u_1^2})^3 \approx \Lambda^{-1} \tag{4.5c}$$

(figure 4*b*).

Also, since these eddies dominate the random flow field, $\overline{u_1^2} \sim u_0'^2$ (see, for example, Hunt, Kaimal & Gaynor 1988).

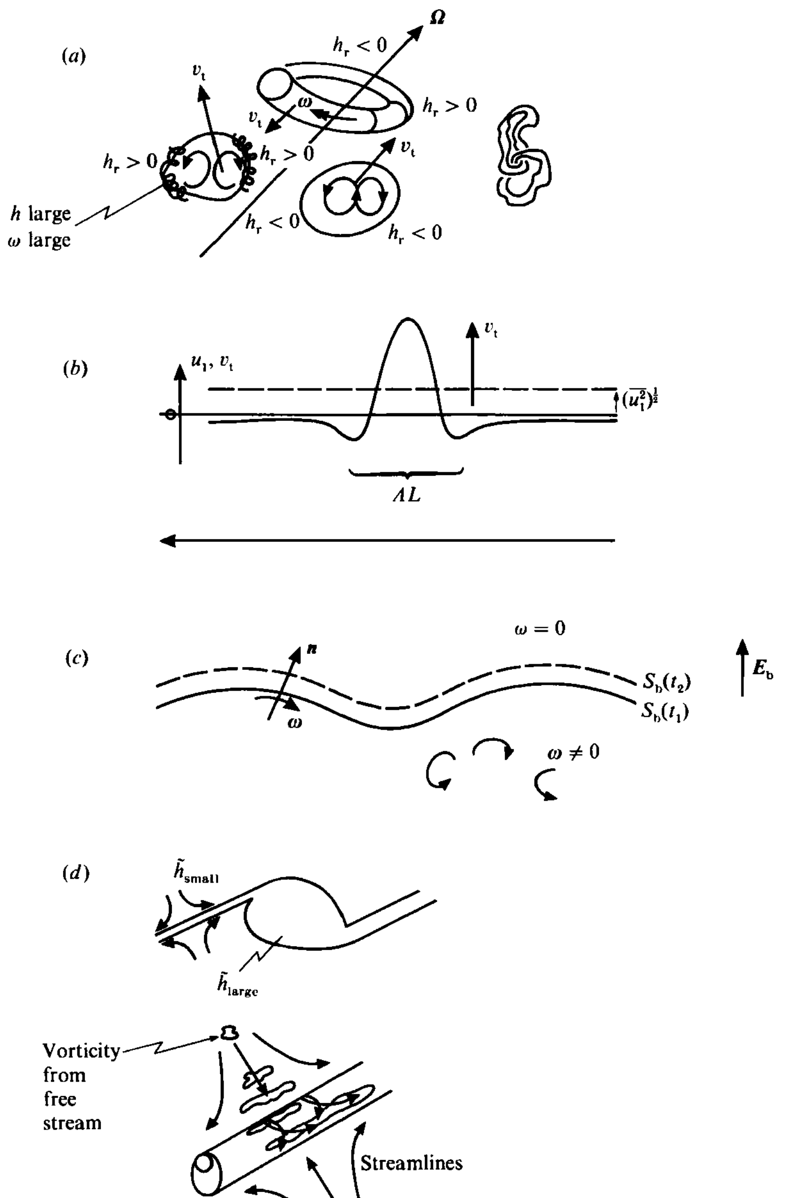


FIGURE 4. (*a*) Helicity fluctuations induced by various eddies moving in a large-scale vorticity field. (*b*) Typical velocity profile associated with an eddy, to show how the number of 'eddies' per unit volume A is related to $\overline{u_1^3}, \overline{u_1^2}$. (*c*) The movement of an interface, at a velocity $\boldsymbol{E}_b$, between a turbulent region ($n < 0$) and a non-turbulent region ($n > 0$). (*d*) The distortion of turbulence at the stagnation points on 'ribs' between vortices, and indications of regions of relatively low- and high-level helicity.

The mean helicity from these random motions $\bar{h}_r$ can be estimated from (3.19). Assuming the eddies are on the average symmetrical, then there is no contribution to $\bar{h}_r$ from the flow external to eddies moving perpendicular to $\boldsymbol{\Omega}$. Part of the contribution to $\bar{h}_r$ comes from the helicity within each eddy H_t, i.e. the integral $\int_{V_t} (\boldsymbol{U}+\boldsymbol{u})\cdot(\boldsymbol{\Omega}+\boldsymbol{\omega})\,\mathrm{d}V$ over the volume V_t of the eddy. The mean over all eddies is

$\{H_t\}$. This produces a contribution to $\bar{h}_t \approx \Lambda\{H_t\}/V_t$ when averaged over all space. The other helicity contribution comes from the motions external to those eddies which have a component of motion parallel or anti-parallel to $\boldsymbol{\Omega}$. Adding the two contributions leads to $\bar{h}_r \approx -C_{H_t}\Lambda(\boldsymbol{\Omega}\cdot\boldsymbol{v}_t)+\bar{h}_t$. Typically the coefficient C_{H_t} varies between $\frac{1}{3}$ and $\frac{1}{2}$ as the eddy geometry varies from elongated to spherical. Using (4.5*b, c*) for estimates of v_t and Λ, it follows that the order of magnitude of the mean helicity produced by the coherent motion is (if $(\overline{u_1^3}) \gtrsim (\overline{u_1^2})^{\frac{3}{2}}$)

$$\bar{h}_r \approx -\frac{(\overline{(\boldsymbol{u}\cdot\boldsymbol{\Omega})^2})^2\cdot\overline{(\boldsymbol{u}\cdot\boldsymbol{\Omega})^3}}{(\overline{(\boldsymbol{u}\cdot\boldsymbol{\Omega})^3})^2}+\bar{h}_t. \tag{4.6}$$

Thus in a rotational flow if non-helical eddies (such as simple vortices and thermals) are generated so as to produce a skewed turbulent velocity field, a mean helicity will in general exist. For example, the external flow around buoyant thermals with a typical velocity w_* rising in a rotating system can contribute a mean helicity proportional to $(\boldsymbol{\Omega}\cdot\boldsymbol{g})\,w_*$. However, within the thermals the helicity is likely to be of opposite sign caused by concentration of vorticity by entrainment. Note that there may be a significant contribution to $\overline{u_i^3}$ from eddies within the inertial subrange, so that eddies with a wide range of scales can contribute to $\bar{h}_r$ (Hunt *et al.* 1988).

Also fluctuations in helicity h'_r are produced by eddies moving either parallel or perpendicular to the mean rotation $\boldsymbol{\Omega}$. The contribution by the flow around the eddies to the normalized intensity of helicity fluctuation is, from (3.17) and (3.19),

$$\tilde{h}'_r \approx \frac{\Lambda\Omega v_t L}{u_0'^2}, \tag{4.7}$$

where L is the scale of these eddies. Using (4.5), it follows that

$$\tilde{h}'_r \approx (\overline{u_1^2}/\overline{u_1^3})\,\Omega L. \tag{4.8}$$

Of course small-scale random wave-like fluctuations also produce helicity fluctuations in rotational flow (Moffatt 1978). But the important physical point of our analysis is that there may be regions of rather large helicity fluctuation just outside eddies or coherent structures moving through a rotational large-scale flow.

In cases where $|(\boldsymbol{u}_c \wedge \boldsymbol{\Omega})|$ is large, regions of relatively high local helicity fluctuation on the sides of eddies are associated with regions of high vortex stretching and larger dissipation. But note that in this situation it is assumed that $|\boldsymbol{\omega}|$ is weak, so that, although $\boldsymbol{\omega}$ and $\boldsymbol{u}$ become nearly parallel in these regions, the normalized fluctuating helicity is small, i.e. $|\tilde{h}'_r| < 1$. If $|\boldsymbol{\omega}|$ were much stronger then the vortices stretched around the eddy would induce velocity fluctuations perpendicular to the vortices which would be stronger than the relative velocity of the eddy v_t. In this case, the helicity would still be small, and the rate of dissipation would be locally large. This limiting case would be consistent with the hypothesis that where the dissipation rate (ϵ) is large, the local helicity is relatively small; but the case of weak vorticity shows that this hypothesis does not imply a monotonic relation between $1/\epsilon$ and h. In fact, our eddy example shows that for weak helicity in straining flows where the helicity rises the dissipation can also increase.

4.2. *Helicity and entrainment*

The main reasons given for considering helicity density have been: that it is interesting because it is a conserved dynamical and topological quantity (for inviscid flows); that is, it may be an indicator of nonlinear interactions; and that it indicates the potential for the generation of magnetic fields in conducting fluids. However, information about helicity may have a more immediate practical use, because it is probably related to entrainment at a boundary between turbulent and non-turbulent motions, such as occurs in jets, wakes, mixing layers, boundary layers, etc.

A surface S of a coherent structure lies between rotational and approximately irrotational flow, and is defined by $S(\boldsymbol{x}_s) = 0$ with unit normal vector $\hat{\boldsymbol{n}}$. Since vorticity is solenoidal and vortex sheets do not exist at such surfaces, the normal component of vorticity must be zero ($\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$) (figure 4$c$). Therefore, $\boldsymbol{\omega}$ is parallel to the surface which implies that $\boldsymbol{\omega}_{rc}$ is also approximately parallel to S, since $|\boldsymbol{\omega}_{rc}|$ is large compared with $|\boldsymbol{\omega}_c|$. However, if the surface moves relative to the local large-scale flow, $\boldsymbol{u}_c$, there must be a component of the velocity fluctuation perpendicular to the surface, i.e. $(\boldsymbol{u}_{rc}\cdot\hat{\boldsymbol{n}}) \neq 0$.

Therefore if $|\boldsymbol{u}_{rc}\cdot\boldsymbol{\omega}_{rc}|$ is small compared with u'^2_{rc}/l, where l is the integral scale of incoherent velocity fluctuations and $u'_{rc} = \{u^2_{rc}\}^{\frac{1}{2}}$, the surface S can move into the irrotational flow around the coherent structure, i.e. the criterion is that $\tilde{h}_{rc} = |\boldsymbol{u}_{rc}\cdot\boldsymbol{\omega}_{rc}|/(u'^2_{rc}/l) \ll 1$. The mean rate of movement of such an interface, in some local frame (e.g. at the edge of a vortex or a jet), is called the boundary entrainment velocity (E_b). If E_b is defined in a frame moving with coherent velocity $\boldsymbol{u}_c$, then it must depend on the modulus of the local helicity of the incoherent turbulence as well as the magnitude of local velocity fluctuations u'_{rc} and on the structure of large scales of motion (being independent of Reynolds number). Thus, where $\tilde{h}_{rc} \sim 1$, E_b/u'_{rc} is small, and where $\tilde{h}_c \ll 1$, $E_b/u'_{rc} \sim 1$. If the turbulence is highly anisotropic with large fluctuations parallel to the surface, so that $\tilde{h}_{rc} \sim 1$, there may still be a significant absolute value of E_b, but the normalized value E_b/u'^2_{rc} will be less than in the case where the velocity fluctuations are more isotropic and $\tilde{h}_{rc} \ll 1$. (The physical meaning and definition of E_b were recently discussed by Turner (1986) and Hunt, Rottman & Britter (1983). Note that E_b may differ in magnitude and sign from other definitions of entrainment velocity, such as the mean velocity (in a fixed frame) induced by the turbulence, or the effective velocity defining the flux of material across a fixed interface in a turbulent flow.)

In the large-scale structures of jets, where $u'_{rc} \sim u'_0$, $\tilde{h}_{rc}$ is small and this is consistent with E_b/u'_0 being of order unity. In the coherent structures measured in mixing layers h'_{rc} has been computed and measured (Hussain 1986) (see figure 4d). At the stagnation point at the centre of the 'ribs' (i.e. longitudinal vortices connecting the larger spanwise vortices), there is irrotational straining by the large-scale motion, transporting external turbulence into the ribs and amplifying the local turbulence. Thus $\{\omega^2_{rc}\}$ and $\{\boldsymbol{u}^2_{rc}\}$ increase, but the helicity $(\boldsymbol{u}_{rc}\cdot\boldsymbol{\omega}_{rc})$ of a fluid element or a fluid volume remains approximately constant during the straining (from (2.4)), so that the normalized helicity $\tilde{h}_{rc}$ is small. In this region the turbulence is strong enough that the boundary entrainment velocity E_b balances the mean straining velocity ($\approx u'_{rc}$) (which opposes the spreading of the turbulent region), i.e. $E_b/u'_{rc} \approx 1$. However, in the rotational regions of the vortex structures, measurements show that $\tilde{h}_{rc}$ is larger, and E_b/u'_{rc} is small.

Maxworthy (1974) shows how the entrainment at the surface of vortex rings is very weak ($E_b/u'_0 \ll 1$), and that the helicity is large. In this case it is associated with wave-like motions along the vortex, in which motions perpendicular to the vortex lines are suppressed.

This hypothesis is consistent with a general dynamical argument. Where the average modulus of normalized helicity is relatively large in a turbulent flow, the production of velocity fluctuations and vorticity diffusion is small, and therefore the entrainment is weak (such as in the 'roller' regions of a mixing layer). On the other hand, where the normalized helicity of small-scale turbulence is weak it is likely to occur in the presence of large-scale straining, because there is stretching of small-scale vorticity and strong velocity fluctuations normal to the small-scale vorticity, such as occurs in the region of low helicity density at the saddle points of mixing layers. Entrainment is large here because these large normal velocity fluctuations diffuse the mean vorticity, and because high gradients of amplified vorticity also amplify its viscous diffusion.

This preliminary examination of fluctuating helicity in coherent structures will be extended.

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